

ON THE SPECIAL \mathbb{A}^n -FIBRATIONS OVER A LINE $\mathbb{A}_{\mathbb{C}}^1$

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ABSTRACT. We have proved the following Problem: *Let R be a \mathbb{C} -affine domain, let T be an element in $R \setminus \mathbb{C}$ and let $i : \mathbb{C}[T] \hookrightarrow R$ be the inclusion. Assume that $R/TR \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ and that $R_T \cong_{\mathbb{C}[T]} \mathbb{C}[T]_T^{[n-1]}$. Then $R \cong_{\mathbb{C}} \mathbb{C}^{[n]}$.* This result leads to the negative solution of the candidate counter-example of V.Arno den Essen : Conjecture E : *Let $A := \mathbb{C}[t, u, x, y, z]$ denote a polynomial ring, and let $f(u) := u^3 - 3u, g(u) := u^4 - 4u^2$ and $h(u) := u^5 - 10u$ be the polynomials in $\mathbb{C}[u]$. Let $D := f'(u)\partial_x + g'(u)\partial_y + h'(u)\partial_z + t\partial_u$ (which is easily seen to be a locally nilpotent derivation on A). Then $A^D \not\cong_{\mathbb{C}} \mathbb{C}^{[4]}$.* Consequently our result in this short paper guarantees that the conjectures : “the Cancellation Problem for affine spaces”, “the Linearization Problem”, “the Embedding Problem” and “the affine \mathbb{A}^n -Fibration Problem” are still open.

1. INTRODUCTION

Let $R^{[n]}$ denote a polynomial ring of n -variables over a ring R .
Let \mathbb{C} be the field of complex numbers.

Our objective is to settle the following Conjecture affirmatively.

Conjecture \mathbf{F}_n . *Let R be a \mathbb{C} -affine domain, let T be an element in $R \setminus \mathbb{C}$ and let $i : \mathbb{C}[T] \hookrightarrow R$ be the inclusion. Assume that $R/TR \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ and that $R_T \cong_{\mathbb{C}[T]} \mathbb{C}[T]_T^{[n-1]}$. Then $R \cong_{\mathbb{C}} \mathbb{C}^{[n]}$.*

By an affine, flat fibration over a scheme S by the *affine n -space* \mathbb{A}^n is meant an affine flat morphism $X \rightarrow S$ of finite type such that for each $s \in S$ the fiber above s is isomorphic to $\mathbb{A}_{k(s)}^n$ over the residue field $k(s)$ of s . Such a fibration is called an *\mathbb{A}^n -bundle over S* if X is locally isomorphic to $\mathbb{A}_S^n := \mathbb{A}^n \times_{\mathbb{Z}} S$ with respect to the Zariski topology on S .

Some studies on \mathbb{A}^1 -fibrations are shown in several articles, for example [10], [11].

Especially, when S is an affine normal \mathbb{C} -curve, the affirmative answer is given in the case $n \leq 3$ as is shown in [10] and [17].

Our result is only a stone on the above case ($n = 4$ and \mathbb{C} -line $\mathbb{A}_{\mathbb{C}}^1$). To tell the truth, our starting plan is not to study \mathbb{A}^1 -fibrations but to crack negatively the candidate counter-example of V.Arno den Essen :

Conjecture E([5]) : Let $A := \mathbb{C}[t, u, x, y, z]$ denote a polynomial ring, and let $f(u) := u^3 - 3u$, $g(u) := u^4 - 4u^2$ and $h(u) := u^5 - 10u$. Let $D := f'(u)\partial_x + g'(u)\partial_y + h'(u)\partial_z + t\partial_u$ (which is easily seen to be a locally nilpotent derivation on A). Then $A^D \not\cong_{\mathbb{C}} \mathbb{C}^{[4]}$.

2. PRELIMINARIES

The following result is well-known.

Lemma 2.1. ([15, (18.4)]) Let R be a ring and R' be an R -algebra such that R' is faithfully flat over R . If I be an ideal of R , then $K(R) \subseteq K(R')$ and $IR' \cap R = I$.

Corollary 2.2. Let R be a ring and R' be an R -algebra such that R' is faithfully flat over R . Let $K(R)$ (resp. $K(R')$) be the total quotient ring of R (resp. R'). Then $IR' \cap K(R) = I$ and $R' \cap K(R) = R$.

Proof. Take $a/b \in K(R) \cap IR'$ with $a, b \in R$, where b is not a zero-divisor in R . Then $a \in bIR' \cap R = bI$ by Lemma 2.7. So $a/b \in I$ and hence $IR' \cap R \subseteq I$. The converse inclusion is trivial. \square

Let $A \hookrightarrow B$ be a ring-extension and let B_p with $p \in \text{Spec}(A)$ denote $S^{-1}B$, where S is a multiplicative set $A \setminus p$.

Proposition 2.3. Let R be a \mathbb{C} -affine domain, let T be an element in $R \setminus \mathbb{C}$ and let $i : \mathbb{C}[T] \hookrightarrow R$ be the inclusion. If $R_T \cong_{\mathbb{C}[T]} \mathbb{C}[T]_T^{[n-1]}$, then $R_{(T-\alpha)\mathbb{C}[T]} \cong_{\mathbb{C}[T]} \mathbb{C}[T]_{(T-\alpha)\mathbb{C}[T]}^{[n-1]}$ and $R/(T-\alpha)R \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ canonically for all $\alpha \in \mathbb{C}^\times$.

Proof. Since $t \in (T-\alpha)\mathbb{C}[T]$ for $\alpha \in \mathbb{C}^\times$, we have $R_{(T-\alpha)\mathbb{C}[T]} = (R_T)_{(T-\alpha)\mathbb{C}[T]} \cong_{\mathbb{C}[T]} (\mathbb{C}[T]_T)_{(T-\alpha)\mathbb{C}[T]}^{[n-1]} = \mathbb{C}[T]_{(T-\alpha)\mathbb{C}[T]}^{[n-1]}$. It is easy to see that $R/(T-\alpha)R = R_{(T-\alpha)\mathbb{C}[T]}/(T-\alpha)R_{(T-\alpha)\mathbb{C}[T]} \cong_{\mathbb{C}} (\mathbb{C}[T]_{(T-\alpha)\mathbb{C}[T]}/(T-\alpha)\mathbb{C}[T]_{(T-\alpha)\mathbb{C}[T]})^{[n-1]} \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ because $T \notin (T-\alpha)\mathbb{C}[T]$ for $\alpha \in \mathbb{C}^\times$. \square

Remark 2.4. We use the same notations as in Proposition 2.3. Since $\mathbb{A}_{\mathbb{C}}^{n-1} \cong_{\mathbb{C}} {}^a i^{-1}(V(T-\alpha)\mathbb{C}[T]) = \text{Spec}(R/(T-\alpha)R)$ for every $\alpha \in \mathbb{C}$, $T-\alpha \in \mathbb{C}[T]$ is a non-zero-divisor on both $\mathbb{C}[T]$ and R , and $R/(T-\alpha)R$ is regular. Hence R is faithfully flat and smooth over $\mathbb{C}[T]$ by [3, Corollaries 6.9 and 6.3], R is smooth over \mathbb{C} and so R is a regular domain by [9, pp.269-270]. It is easy to see that R is a UFD by Nagata's Theorem [7, Theorem 7.1] because T is a prime element in R and $R_T \cong_{\mathbb{C}[T]} \mathbb{C}[T]_T^{[n-1]}$ is a UFD.

Definition 2.5. [13, (1.L)] Let A be a ring, $A \neq 0$. The Jacobson radical of A , $\text{rad}(A)$, is the intersection of all maximal ideals of A .

Remark 2.6. [13, (1.L)] Let A be a ring, $A \neq 0$ and let I be an ideal of A . If $1+x$ is a unit for each $x \in I$, then $I \subseteq \text{rad}(A)$.

Lemma 2.7. [13, (24.A)] Let A be a Noetherian ring with an I -adic topology, where I is an ideal of A . Then the following are equivalent:

- (1) A is a Zariski ring, that is, every ideal is closed in A ;
- (2) $I \subseteq \text{rad}(A)$;
- (3) the completion \widehat{A} of A is faithfully flat over A .

3. THE AFFIRMATIVE ANSWER TO CONJECTURE F_n

In this section, we show the following, which implies the conjecture F_n is affirmative for all $n \in \mathbb{N}$.

Theorem 3.1. Let R be a \mathbb{C} -affine domain with $R^\times = \mathbb{C}^\times$, let T be an element in $R \setminus \mathbb{C}$ and let $i : \mathbb{C}[T] \hookrightarrow R$ be the inclusion. Assume that $R/TR \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ and that $R_T \cong_{\mathbb{C}[T]} \mathbb{C}[T]_T^{[n-1]}$. Then $R \cong_{\mathbb{C}} \mathbb{C}^{[n]}$.

Proof. Let $p_\alpha = (T - \alpha)\mathbb{C}[T]$ denote a prime ideal of $\mathbb{C}[T]$ for $\alpha \in \mathbb{C}$. By Remark 2.4, R is faithfully flat and smooth over $\mathbb{C}[T]$ and R is a regular UFD. Let $V_\alpha := \mathbb{C}[T]_{p_\alpha}$.

Let the canonical homomorphism $\sigma : R \rightarrow R/TR = \mathbb{C}[\overline{Z}_1, \dots, \overline{Z}_{n-1}]$, a polynomial ring over \mathbb{C} . Take $Z_i \in R$ such that $\sigma(Z_i) = \overline{Z}_i$ ($1 \leq i \leq n-1$), and put $A := \mathbb{C}[T][Z_1, \dots, Z_{n-1}]$, which is a polynomial ring over $\mathbb{C}[T]$ (over \mathbb{C}).

Let $A^\#$ denote the integral closure of A in $K(R)$. Then $A^\#$ is a finite A -algebra and hence a \mathbb{C} -affine domain, which is well-known. Since R is regular by Remark 2.4, $A^\# \subseteq R$ and $K(R) = K(A^\#)$.

Set

$$\begin{aligned} p &:= p_0 = T\mathbb{C}[T] \\ V &:= \mathbb{C}[T]_p \\ A' &:= A_{TA} \text{ (with the } pA' \text{-adic topology)} \\ A^* &:= A^\# \otimes_A A' \text{ (with the } pA^* \text{-adic topology)} \\ R' &:= R_{TR} \text{ (with the } pR' \text{-adic topology)} \\ \widehat{A'} &:= \text{(the } pA' \text{-adic completion of } A') \\ \widehat{A^*} &:= \text{(the } pA^* \text{-adic completion of } A^*) \\ \widehat{R'} &:= \text{(the } pR' \text{-adic completion of } R'). \end{aligned}$$

Note that p is a prime in $\mathbb{C}[T]$ and that T is a prime element both in R and A . It follows that pA and pR are prime, respectively, and both $\mathbb{C}[T]_{(T)} \hookrightarrow A'$ and $\mathbb{C}[T]_{(T)} \hookrightarrow R'$ are flat by Remark 2.4. Note that A^* is finite over A' , that $A' \subseteq A^* \subseteq R'$ and that $\widehat{A^*} = A^* \otimes_{A'} \widehat{A'}$ by [1, Proposition 10.13].

Let $\varphi : A \hookrightarrow R$ and $\varphi' : A' \hookrightarrow R'$ denote the inclusions.

We can obviously assert that the Jacobson radicals of A' , A^* and R' contains the ideal TA' , TA^* and TR' by Remark 2.6, respectively, which means these topological

rings are Zariski rings. So $\widehat{A'}$ (resp. $\widehat{A^*}, \widehat{R'}$) is faithfully flat over A' (resp. A^*, R') by Lemma 2.7.

Hence $A' \otimes_V (p^r/p^{r+1}) = p^r A'/p^{r+1} A'$ and $R' \otimes_V (p^r/p^{r+1}) = p^r R'/p^{r+1} R'$ ($r \in \mathbb{N}$) by the faithfully flatness of A' and R' over V . We obtain the associated graded rings as follows:

$$\begin{array}{ccc} gr_{pA'}(A') & = & A'/pA' \oplus pA'/p^2A' \oplus \cdots \oplus p^r A'/p^{r+1} A' \oplus \cdots \\ \downarrow gr(\varphi'_r) & & \downarrow gr(\varphi'_r) \\ gr_{pR'}(R') & = & R'/pR' \oplus pR'/p^2R' \oplus \cdots \oplus p^r R'/p^{r+1} R' \oplus \cdots \end{array}$$

Then $gr(\varphi'_r)$ is injective because $p^{r+1}R \cap A = p^{r+1}A$ by the faithfully flatness of $A \hookrightarrow R$. Moreover we have $p^r A/p^{r+1} A = p^r A \otimes_A A/pA = p^r \otimes_{\mathbb{C}[T]} A/pA \cong p^r \otimes_{\mathbb{C}[T]} R/pR = p^r R \otimes_R R/pR = p^r R/p^{r+1} R$ implies $gr(\varphi'_r) : p^r A'/p^{r+1} A' \cong p^r R'/p^{r+1} R'$ by applying $- \otimes_{\mathbb{C}[T]} V$. We have $gr_{pA'}(A') = gr_{pR'}(R')$. Therefore

$$\widehat{\varphi}' : \widehat{A'} \longrightarrow \widehat{R'}$$

is bijective by [1, Lemma 10.23].

The inclusion $A' \hookrightarrow A^*$ and the flatness of $A' \rightarrow \widehat{A'}$ induce

$$\widehat{R'} = \widehat{A'} = A' \otimes_{A'} \widehat{A'} \hookrightarrow A^* \otimes_{A'} \widehat{A'} = \widehat{A^*}$$

because $\widehat{R'} = \widehat{A'}$ as was shown above. That is, $\widehat{R'} \hookrightarrow \widehat{A^*}$. Since $K(A^*) = K(R')$, we conclude that

$$R' = \widehat{R'} \cap K(R') \hookrightarrow \widehat{A^*} \cap K(A^*) = A^*$$

by Corollary 2.2 and hence that $R' = A^*$ because $A^* \subseteq R'$ trivially. Then $R' = A^*$ is (module-)finite over A' , which implies that $A' \otimes_{A'} \widehat{A'} = \widehat{A'} = \widehat{R'} = R' \otimes_{A'} \widehat{A'}$. Since $A' \hookrightarrow \widehat{A'}$ is faithfully flat, we conclude that $A' = R'$, that is $R_{TR} = A_{TA}$. Hence $K(A) = K(R)$.

We see that A and R are UFDs on $K(A) = K(R)$, noting that R is a UFD (Remark 2.4). Since $A \subseteq R$, we have $\text{Ht}_1(R) \cap A \subseteq \text{Ht}_1(A)$, where $\text{Ht}_1(R) \cap A = \{P \cap A \mid P \in \text{Ht}_1(R)\}$. Take a prime element $a \in A$, then $aR \subseteq P$ for some $P \in \text{Ht}_1(R)$ because $R^\times = \mathbb{C}^\times$. It follows that $\text{Ht}_1(A) \subseteq \text{Ht}_1(R) \cap A$ and consequently, $\text{Ht}_1(R) \cap A = \text{Ht}_1(A)$. Therefore we have

$$A = \bigcap_{q \in \text{Ht}_1(A)} A_q = \bigcap_{Q \in \text{Ht}_1(R)} R_Q = R.$$

Hence $R = B \cong_{\mathbb{C}} \mathbb{C}^{[n]}$. □

As a special case, we have

Corollary 3.2. *Let R be a \mathbb{C} -affine domain, let T be an element in $R \setminus \mathbb{C}$ and let $i : \mathbb{C}[T] \hookrightarrow R$ be the inclusion. Assume that $R/TR \cong_{\mathbb{C}} \mathbb{C}^{[4]}$ and that $R_T \cong_{\mathbb{C}[T]} \mathbb{C}[T]^{[3]}$. Then $R \cong_{\mathbb{C}} \mathbb{C}^{[4]}$.*

4. AN ADDITIONAL RESULT

Arno van den Essen [5, **Conjecture 5.1**] was inspired by Shastri's embedding ([16]), and posed the following conjecture. If this conjecture E were affirmative, then “The Cancellation Problem for Affine Spaces(the Zariski Problem)” had a negative solution in the case $n = 5$.

Conjecture E *Let $A := \mathbb{C}[t, u, x, y, z]$ denote a polynomial ring, and let $f(u) := u^3 - 3u$, $g(u) := u^4 - 4u^2$ and $h(u) := u^5 - 10u$. Let $D := f'(u)\partial_x + g'(u)\partial_y + h'(u)\partial_z + t\partial_u$ (which is easily seen to be a locally nilpotent derivation on A). Then $A^D \not\cong_{\mathbb{C}} \mathbb{C}^{[4]}$.*

Conjecture E asserted possibly that $A^D[X] \cong \mathbb{C}^{[5]}$ but $A^D \not\cong \mathbb{C}^{[4]}$.

Therefore we can complete Hypothesis F_4 in the following preprint, in which we assumed Corollary 3.2 as the hypothesis:

“Note on the candidate counter-example in the cancellation problem for affine spaces”(which has been submitted to a certain Journal).

We finally have finished the counter-argument to the candidate counter-example conjecture E in [5] mentioned in the introduction, that is, the following result holds.

Theorem 4.1. *Let $A := \mathbb{C}[t, u, x, y, z]$ denote a polynomial ring, and let $f(u), g(u)$ and $h(u)$ be the polynomials defined above. Let $D := f'(u)\partial_x + g'(u)\partial_y + h'(u)\partial_z + t\partial_u$ (which is easily seen to be a locally nilpotent derivation on A). Then $A^D \cong_{\mathbb{C}} \mathbb{C}^{[4]}$.*

Consequently our result in this short paper guarantees that the conjectures : the Cancellation Problem for affine spaces”, “the Linearization Problem”, “the Embedding Problem” and “the affine \mathbb{A}^n -Fibration Problem” are still open.

REFERENCES

- [1] M.F.Atiyah and L.G.MacDonald, Introduction to Commutative Algebra, Addison-Wesley, London (1969).
- [2] A.Altman and S.Kleiman, Introduction to Grothendieck duality theory, Lecture Notes in Math. 146, Springer-Verlag (1970).
- [3] D.Eisenbud, Commutative Algebra with a view toward algebraic geometry, GTM 150, Springer-Verlag (1995).
- [4] H.Bass, E.H.Connel and D.L.Wright, Locally polynomial algebras are symmetric algebras, Bull. Amer. Math. Soc., Vol. 82 (1976), 719-720.
- [5] A.van den Essen and P.van Rossum, Triangular derivations related to problems on affine n -space, Proc. Amer. Math. Sc. 130 (2002), 1311-1322.
- [6] A. van den Essen, Polynomial Automorphisms and the Jacobian Conjecture, volume 190 of Progress in Mathematics, Birkhäuser-Verlag (2000).
- [7] R.Fossum, The Divisor Class Group of a Krull Domain, Springer-Verlag (1973).
- [8] T.Fujita, On Zariski Problem, Proc. Japan Acad. Ser. A Math. Sci 55 (1975), 106-110.
- [9] R.Hartshorne, Algebraic Geometry, GTM 52, Springer-Verlag (1977).
- [10] T.Kambayashi, On one-parameter family of affine planes, Invent. Math. no. 52 (1979), 275-281.
- [11] T.Kambayashi and D.Wright, Flat families of affine lines are affine-line bundles, Illinois Journal of Math. Vol. 29 (1985), 672-681.

- [12] E.Kunz, Introduction to Commutative Algebra and Algebraic Geometry, Birkhäuser (1985).
- [13] H.Matsumura, Commutative Algebra, Benjamin, New York (1970).
- [14] H.Matsumura, Commutative Ring Theory, Cambridge Univ. Press (1980).
- [15] M.Nagata, Local Rings, Interscience, 1962.
- [16] A.Shastri, Polynomial representations of knots, Tohoku Math. J. (1992), 11-17.
- [17] A.Sathaye, Polynomial rings in two variables over a D.V.R.: A Criterion, Invent. Math., 74 (1983) 159-168,

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